

# Novel considerations about the non-equilibrium regime of the tricritical point in a metamagnetic model: localization and tricritical exponents

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## Abstract

We have investigated the time-dependent regime of a two-dimensional metamagnetic model at its tricritical point via Monte Carlo simulations. First of all, we obtained the temperature and magnetic field corresponding to the tricritical point of the model by using a refinement process based on optimization of the coefficient of determination in the log-log fit of magnetization decay as function of time. With these estimates in hand, we obtained the dynamic tricritical exponents  $\theta$  and  $z$  and the static tricritical exponents  $\nu$  and  $\beta$  by using the universal power-law scaling relations for the staggered magnetization and its moments at early stage of the dynamic evolution. Our results at tricritical point confirm that this model belongs to the two-dimensional Blume-Capel model universality class for both static and dynamic behaviors, and also they corroborate the conjecture of Janssen and Oerding for the dynamics of tricritical points.

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In the study of phase transitions and critical phenomena, systems which exhibit multicritical behavior have been the subject of a great number of works. Theoretically, the tricritical phase transition of the Blume-Capel [1] model is one of the most studied. However, there are other models showing the existence of such multicritical points, for instance, the metamagnetic model [2], the Blume-Capel model with antiferromagnetic exchange interaction and external magnetic field added [3], and the random-field Ising model [4]. In order to investigate these phenomena, several techniques have been employed, including series expansions [5], linked-cluster expansion [6], mean-field theory [7], renormalization group [8, 9, 10, 11], transfer matrix [12, 13, 14, 15], Monte Carlo simulations [16, 17, 18, 19], and Monte Carlo renormalization group methods [20, 21, 22]. Experimentally, the phase transitions of metamagnetic systems such as in the compound FeBr<sub>2</sub> [23, 24] have also been studied in order to understand the tricritical behavior that appears as a consequence of a competition between the antiferromagnetic and ferromagnetic

coupling constants present in this magnetic system.

The two-dimensional spin- $\frac{1}{2}$  metamagnetic model is defined by the Hamiltonian

$$\mathcal{H} = J_1 \sum_{nn} \sigma_i \sigma_j - J_2 \sum_{nmn} \sigma_i \sigma_k + H \sum_i \sigma_i \quad (1)$$

where  $J_1, J_2 > 0$  and  $\sigma_i = \pm 1$  are the spin variables. The considered model has two sublattices where the first sum extends over all nearest-neighbor pairs (intersublattice) and second one over all next-nearest neighbor pairs (intra-sublattice), respectively. The parameters  $J_1$  and  $J_2$  are the antiferromagnetic and ferromagnetic coupling constants, respectively, and  $H$  is the external magnetic field.

The order parameter of the model is the staggered magnetization, conveniently defined by

$$M(t) = \frac{1}{N} \sum_{i=1}^L \sum_{j=1}^L (-1)^{i+j} \sigma_{i,j} = M_1(t) - M_2(t), \quad (2)$$

where  $N = L^2$ ,  $L$  is the linear size of the square lattice. Here  $M_1(t) = \frac{2}{N} \sum_{i=1}^L \sum_{j=1}^L \sigma_{i,j} \delta_{\text{mod}(i+j,2),0}$  and  $M_2(t) = \frac{2}{N} \sum_{i=1}^L \sum_{j=1}^L \sigma_{i,j} \delta_{\text{mod}(i+j,2),1}$  denote the magnetizations of the respective sublattices. This definition shows that there is an inversion of the meaning of ordered and disordered state. In order to obtain an ordered state, it is necessary to occupy the sites of the lattice with spins  $+1$  ( $-1$ ) where the sum  $i+j$  is odd (even), or vice-versa. On the other hand, the null magnetization may be obtained when all sites are occupied with the spins of the same kind.

On the contrary of the Blume-Capel model, the phase diagram of the metamagnetic model has not yet been completely understood. This is due to the controversial results between the experimental and theoretical works concerning the phase transitions of the system. If on the one hand, this model exhibits a rich phase diagram in the temperature-field plane with a line of second-order phase transitions, a line of first-order phase transitions and a tricritical point which is located at the point where the first and second order transition lines join each other with the same slope, on the other hand the mean-field theory [25] predicts that such tricritical point depends on the value of the ratio between the coupling constants. They only predicted the existence of a tricritical point for  $R = J_2/J_1 > 3/5$ , while for  $R < 3/5$  in the mean field approximation the model exhibits two Ising-like critical points: a critical endpoint corresponding to a point that ends at the first order line coming from the second order line and a double critical endpoint (bicritical) that corresponds to the terminal point of the first order transition line. Although for the three-dimensional metamagnetic model Herrmann et al. [26] showed via Monte Carlo Renormalization group that such critical endpoints exist, experimental works have not found those points in any real metamagnetic system, and also there is no evidence of such points for the two-dimensional metamagnetic systems as verified in different works (see for example [13], [27]). Similarly, Santos e Figueiredo [28] by using master equation formalism on the context of dynamical pair approximation, also in two dimensions, did not find any evidence for the decomposition of the tricritical point into the critical and bicritical end points as predicted by the mean field theory. More recently, other authors exclude the possibility of existence of these two critical endpoints even for three dimensions – Geng et al. [29], by using effective field theory, showed that there is no fourth-order critical point or reentrant phenomenon in the phase diagram. Finally, other authors [30] by performing MC simulations showed that there is no evidence of such a decomposition in a critical endpoint and a bicritical endpoint and such simulations produce a tricritical behaviour even for a coupling ratio as small as  $R = 0.01$ .

Although the previous estimates of the critical exponents for this model support the assertion that it belongs to the same universality class of the Blume-Capel model, the nonequilibrium critical behavior of this system has not been completely investigated up to date. Santos and Figueiredo [31] studied a similar layered metamagnetic model far from equilibrium by using short-time Monte Carlo simulations. They estimated the static critical exponents  $\beta$  and  $\nu$  and

the dynamic critical exponent  $z$  on the continuous transition line, but the tricritical exponents were not obtained. They also showed that although the critical exponent  $\nu$  remains the same along the continuous transition line, the exponent  $\beta$  departs from the expected value as we approach the tricritical point of the model.

The study of the dynamic critical properties of statistical systems has been a subject of considerable interest in non-equilibrium physics after the works by Janssen, Schaub and Schmittmann [32], and Huse [33]. By using, respectively, renormalization group techniques and numerical calculations, they showed that universality and scaling behavior are already present in systems since their early stages of the time evolution after quenching from high temperatures to the critical one. As a result, the study of the critical properties of statistical systems became in some sense simpler, because they allow to circumvent the well-known problem of critical slowing down, characteristic of the long-time regime.

The dynamic scaling relation obtained by Janssen *et al.* for the  $k$ -th moment of the magnetization  $M$ , extended to systems of finite size [34, 35], is written as

$$\langle M^k \rangle(t, \tau, L, m_0) = b^{-k\beta/\nu} \langle M^k \rangle(b^{-z}t, b^{1/\nu}\tau, b^{-1}L, b^{x_0}m_0), \quad (3)$$

where  $t$  is the time,  $b$  is an arbitrary spatial rescaling factor,  $\tau = (T - T_c)/T_c$  is the reduced temperature and  $L$  is the linear size of the lattice. Here, the operator  $\langle \dots \rangle$  denotes averages over different configurations due to different possible time evolution from each initial configuration compatible with a given initial magnetization  $m_0$ . The exponents  $\beta$  and  $\nu$  are the equilibrium critical exponents associated to the order parameter and the correlation length respectively, and  $z$  is the dynamic exponent characterizing time correlations at equilibrium.

After choosing the scaling  $b^{-1}L = 1$  at the  $T = T_c$  ( $\tau = 0$ ), and  $k = 1$ , we obtain  $\langle M \rangle(t, L, m_0) = L^{-\beta/\nu} \langle M \rangle(L^{-z}t, L^{x_0}m_0)$ . Denoting  $u = tL^{-z}$  and  $w = L^{x_0}m_0$ , one has:  $\langle M \rangle(u, w) = \langle M \rangle(L^{-z}t, L^{x_0}m_0)$ . The derivative with respect to  $L$  is:

$$\begin{aligned} \frac{\partial \langle M \rangle}{\partial L} &= (-\beta/\nu)L^{-\beta/\nu-1} \langle M \rangle(u, w) + \\ &L^{-\beta/\nu} \left[ \frac{\partial \langle M \rangle}{\partial u} \frac{\partial u}{\partial L} + \frac{\partial \langle M \rangle}{\partial w} \frac{\partial w}{\partial L} \right], \end{aligned}$$

where  $\partial u / \partial L = -ztL^{-z-1}$  and  $\partial w / \partial L = x_0 m_0 L^{x_0-1}$ . In the limit  $L \rightarrow \infty$ ,  $\partial_L \langle M \rangle \rightarrow 0$ , one has:  $x_0 w \frac{\partial \langle M \rangle}{\partial w} - zu \frac{\partial \langle M \rangle}{\partial u} - \beta/\nu \langle M \rangle = 0$ . The separability of the variables  $u$  and  $w$  in  $\langle M \rangle(u, w) = M_1(u)M_2(w)$  leads to  $x_0 w M_2' / M_2 = \beta/\nu + zu M_1' / M_2$ , where the prime means the derivative with respect to the argument. Since the left-hand side of this equation depends only on  $w$  and the right-hand side depends only on  $u$ , they must be equal to a constant  $c$ . Thus,  $M_1(u) = u^{(c/z) - \beta/(\nu z)}$  and  $M_2(w) = w^{c/x_0}$ , resulting in  $\langle M \rangle(u, w) = m_0^{c/x_0} L^{\beta/\nu} t^{(c-\beta/\nu)/z}$ . Returning to the original variables, one has:  $\langle M \rangle(t, L, m_0) = m_0^{c/x_0} t^{(c-\beta/\nu)/z}$ .

On one hand, choosing  $c = x_0$  and denoting  $\theta = (x_0 - \beta/\nu)/z$ , at criticality ( $\tau = 0$ ), we obtain the algebraically behavior of the magnetization,

$$\langle M \rangle(t) \sim m_0 t^\theta. \quad (4)$$

This can be observed by a finite time scaling  $b = t^{1/z}$ , Eq. (3), at critical temperature ( $\tau = 0$ ), which leads to  $\langle M \rangle(t, m_0) = t^{-\beta/(\nu z)} \langle M \rangle(1, t^{x_0/z} m_0)$ . Defining  $x = t^{x_0/z} m_0$ , an expansion of the averaged magnetization around  $x = 0$  results in:  $\langle M \rangle(1, x) = \langle M \rangle(1, 0) + \partial_x \langle M \rangle|_{x=0} x + \mathcal{O}(x^2)$ . By construction  $\langle M \rangle(1, 0) = 0$ , since  $x = t^{x_0/z} m_0 \ll 1$  and  $\partial_x \langle M \rangle|_{x=0}$  is a constant. Discarding the quadratic terms we obtain the expected power law behavior  $\langle M \rangle_{m_0} \sim m_0 t^\theta$ , which is valid only for a characteristic time scale  $t < t_{\max} \sim m_0^{-z/x_0}$ .

Then, in this new universal regime, in addition to the familiar set of critical exponents described above, a new dynamic critical exponent  $\theta$  is found. This exponent, independent of the previously known ones, characterizes the so called ‘‘critical initial slip’’, the anomalous behavior of the magnetization when the system is quenched to the critical

temperature  $T_c$ . In addition, a new critical exponent  $x_0$  is introduced to describe the dependence of the scaling behavior on the initial conditions. This exponent represents the anomalous dimension of the initial magnetization  $m_0$  and is related to the exponent  $\theta$  as  $x_0 = \theta z + \beta/\nu$ .

On the other hand, the choice  $c = 0$  corresponds to the case which the system does not depend on the initial trace and in which  $m_0 = 1$  leads to simple power law:

$$\langle M \rangle_{m_0=1} \sim t^{-\beta/(\nu z)}, \quad (5)$$

which corresponds to decay of magnetization at long times ( $t > t_{\max}$ ).

Unlike the second-order phase transition, the behavior of a thermodynamic system is more complex at a tricritical point and the corresponding exponent  $\theta$  may assume negative values. This assertion was theoretically deduced by Janssen and Oerding [36] and numerically confirmed by da Silva *et al.* [37] through short-time Monte Carlo simulations at the tricritical point of the Blume-Capel model. However, as shown by some researchers, negative values of the exponent  $\theta$  can be also found in systems exhibiting continuous phase transitions, as for instance, the Baxter-Wu [38, 39], multispin [40], and 4-state Potts models [41].

At a tricritical point, the magnetization shows a crossover from the logarithmic behavior  $M(t) \sim m_0 [\ln(t/t_0)]^{-a}$  at short times  $t \ll m_0^{-4}$  to  $t^{-1/4}$  power law with logarithmic corrections,  $M(t) \sim [t/\ln(t/t_0)]^{-1/4}$  in three dimensions. The above behavior can be stated in the generalized form [36]

$$M(t) = m_0 \left[ \ln \left( \frac{t}{t_0} \right) \right]^{-a} F_M(x), \quad (6)$$

where

$$x = \left\{ \left( \frac{t}{\ln(t/t_0)} \right)^{\frac{1}{4}} \left[ \ln \left( \frac{t}{t_0} \right) \right]^{-a} m_0 \right\}. \quad (7)$$

In Eq. (6) the function behave as  $F_M(x) \sim 1$  or  $F_M(x) \sim 1/x$  for vanishing and large arguments, respectively. Below three dimensions it reduces to the scaling form given by Eq. (4), but now the exponent  $\theta$  is the exponent related to the tricritical point of the relaxation process at early times.

In the present work, the simulations were carried out for square lattices with linear dimension  $L = 160$  and periodic boundary conditions for all performed experiments. The estimates for each exponent were obtained from five independent bins at the tricritical point, each one consisting of  $N_{run}$  (number of different time series of magnetization or its upper moments from an initial configuration) runs and  $N_{MC}$  Monte Carlo sweeps. The error bars are fluctuations of the averages obtained from those bins and the dynamic evolution of the spins is local and updated by the heat-bath algorithm. Here it is important to mention that finite size scaling effects are negligencible for the considered size  $L = 160$ . For example, by using the Eq. (5) with  $N_{run} = 10000$  runs and  $L = 80$ , we obtained  $\beta/(\nu z) = 0.03938(5)$  from the time interval  $[80, 300]$  while for  $L = 140$  we obtained  $\beta/(\nu z) = 0.03934(5)$  for the same interval and number of runs. So, within statistical errors we can not distinguish the results for  $L = 80$  and  $L = 140$ . Then, we are very comfortable with  $L = 160$ . Other details about simulations will be supplied according to development of this manuscript.

In this paper, we performed time-dependent Monte Carlo (MC) simulations to explore the tricritical behavior of the two-dimensional metamagnetic model. First of all, we worked on localization of the tricritical point by using a recent refinement process developed by da Silva *et al.* [42]. By considering as input the parameter  $\alpha = J_2/J_1 = 1/2$ , the supposed tricritical temperature  $T_i$ , the resolution  $\Delta H$ ,  $N_{MC} = 150$  and  $N_{run} = 1000$  (a large number of runs is not required to these experiments since relaxation from ordered initial lattices are very stable, differently from evolutions from disordered initial states which demand a lot of runs), we performed MC simulations starting always from the

ordered state ( $m_0 = 1$ ) in order to estimate the value of the external magnetic field  $H_t$  at the tricritical point. With the tricritical set  $T_t$  and  $H_t$  in hand, we are then able to estimate some dynamic and static critical exponents of the proposed model.

To reach our goal, the time evolution of the magnetization (Eq. (5)) is obtained for each value of the external magnetic field  $H_i$  in a range  $[H_{\min}, H_{\max}]$ , where  $H_i = H_{\min} + i \cdot \Delta H$ ,  $i = 0, \dots, n$ , and  $n = (H_{\max} - H_{\min})/\Delta H$ . Then, the  $H_t$  is obtained by using the so-called determination coefficient of the fit

$$r = \frac{\sum_{t=1}^{N_{MC}} (\overline{\ln \langle M \rangle} - a - b \ln t)^2}{\sum_{t=1}^{N_{MC}} (\overline{\ln \langle M \rangle} - \ln \langle M \rangle(t))^2}. \quad (8)$$

Here, the closer to the unity is the coefficient, the better is the fit and the estimate of  $H_t$ . This approach is simpler to calculate than other schemes, such as the goodness of fit, for example.

In Eq. (8)  $\overline{\ln \langle M \rangle} = (1/N_{MC}) \sum_{t=1}^{N_{MC}} \ln \langle M \rangle(t)$ ,  $\langle M \rangle(t) = (1/L^2) \sum_{j=1}^{N_{run}} M_j(t)$ , with  $M_j(t)$  denoting the magnetization of  $j$ -th run of  $t$ -th MC step,  $a$  and  $b$  are the linear coefficient and the slope in the linear fit  $\ln \langle M \rangle$  versus  $\ln t$ , respectively. In our experiments we discarded the initial 30 MC steps for better estimate. In previous work (Ref. [42, 43]) we performed two successive refinements: one with larger  $\Delta^{(0)}$  and another more refined (smaller)  $\Delta^{(1)}$ . Here since we have a good initial kick, we performed only one refinement with  $\Delta H = 10^{-3}$ .

In the first attempt to obtain our set of tricritical parameters ( $T_t, H_t$ ), we considered the results obtained by Landau and Swendsen [20],  $T_t = 1.208(9)$  and  $H_t = 3.965(17)$ , which were obtained through MC renormalization techniques. So, we fixed  $T_t = 1.208$  and changed  $H_t$  in order to obtain its best value through Eq. (8). Figure 1 shows the plot of  $r$  versus  $H$  when  $H_{\min} = 3.9$  and  $H_{\max} = 4.0$ .

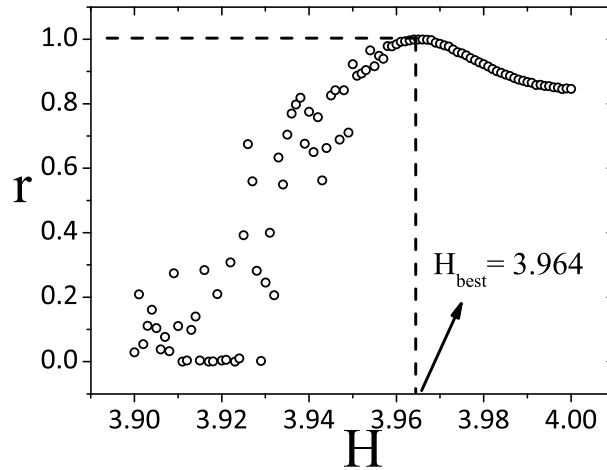


Figure 1: Determination coefficient  $r$  as function of  $H_t$  from  $H_{\min} = 3.9$  up to  $H_{\max} = 4.0$  with  $\Delta H = 0.001$ , for the fixed temperature  $T_t = 1.208$ .

The best value corresponding to  $T_t = 1.208$  found by our refinement is  $H_t = H_{best} = 3.964$  which corroborates the value found in literature. Since we have observed that our non-equilibrium method is able to determine the tricritical point, we can now use this estimate to check if, for instance, the exponent  $z$  is consistent with the results of literature.

For this purpose we used a function  $F_2(t)$  given by [44]

$$F_2(t) = \frac{\langle M^2(t) \rangle_{m_0=0}}{\langle M(t) \rangle_{m_0=1}^2} \sim t^{d/z}, \quad (9)$$

where  $d$  is the dimension of the system. This approach, that mixes moments of magnetization under different initial conditions, has proved to be very efficient in estimating the exponent  $z$  for a great number of models [37, 38, 45, 46, 47]. Here it is important to mention that the calculation of the second moment  $\langle M^2(t) \rangle_{m_0=0}$  must be performed with initial magnetization per spin  $m_0 = 0$ . However, the initial configuration of the system must be chosen at random instead of ordered one. As mentioned above, from Eq. (2), an initial ordered configuration ( $\sigma_{i,j} = 1$  for all sites) gives  $M(0) = m_0 = 0$  and, although this is the simpler way, it is a very correlated one. Nevertheless, the sharp preparation of random initial condition with  $m_0 = 0$  is also straightforward performed: we distribute randomly, and with same probability, spins  $+1$  and  $-1$  on the lattice sites. Then, an adjustment process is performed: If the magnetization is negative, we choose randomly one site  $(i, j)$ , and while the magnetization remains negative, we flip  $+ \rightarrow -$  whether  $i + j$  is odd and  $- \rightarrow +$  otherwise. If the magnetization is positive we make exactly the opposite: we flip  $- \rightarrow +$  if  $i + j$  is odd and  $+ \rightarrow -$  otherwise. This process is done until the magnetization vanishes.

On the other hand, to obtain the magnetization  $\langle M(t) \rangle$ , we must perform simulations with ordered initial configurations which are trivially prepared by putting in a site  $(i, j)$  a spin  $+1$  if  $i + j$  is even and  $-1$  otherwise.

In this paper, we used for computation of averaged time series of the  $k$ -th moment of magnetization, i.e.,  $\langle M^k(t) \rangle_{m_0} \times t$ , a total of  $N_{run}$  runs that depends on  $m_0$  considered. The error bars were obtained from  $N_b$  different bins (of course each bin means the quantities – magnetization or their moments – were averaged over the  $N_{run}$  time series). In order to obtain the tricritical exponents, we used always  $N_b = 5$ ,  $N_{run} = 20000$  for experiments that require disordered initial configurations, such as those ones to obtain the exponents  $\theta$  and  $z$  (small or null values of  $m_0$ , respectively), and  $N_{run} = 10000$  for experiments that demand ordered initial configurations, such as those ones to estimate the exponents  $z$ ,  $\beta$  and  $\nu$ . Our results in the plots correspond to more refined estimate  $\overline{\langle M^k(t) \rangle} = (1/N_b) \sum_{i=1}^{N_b} \langle M^k(t) \rangle^{(i)}$  and the error bars (standard deviation of average) were estimated as  $\sigma/\sqrt{N_b} = \left( \frac{1}{N_b(N_b-1)} \sum_{i=1}^{N_b} \left[ \langle M^k(t) \rangle^{(i)} - \overline{\langle M^k(t) \rangle} \right]^2 \right)^{1/2}$ , where  $\langle M^k(t) \rangle^{(i)}$  denotes the average of  $k$ -th moment of magnetization of the  $i$ -th bin.

In Figure 2 (a), we showed the time evolution of  $F_2(t)$  in a log-log plot: the black squares correspond to simulations performed with estimates  $T_t = 1.208$  and  $H_t = 3.965$  obtained in [20]. Since  $F_2(t)$  is obtained from two different time evolutions, we obtain the exponent  $z$  by making a crossover of a bin of  $\langle M^2(t) \rangle_{m_0=0}$  and another bin of  $\langle M(t) \rangle_{m_0=1}$ , resulting in  $N_b = 25$  and not simply  $N_b = 5$ .

Then we built a simple algorithm that, for different peaces of time evolution, performs a linear fit and always keeps the same number of points. We used for such estimates a maximum number of MC steps  $N_{MC} = 400$ . Our algorithm supplies as output the peace of the time window  $[t_{\min}, t_{\max}]$  corresponding to the best goodness of fit [48], as well as the value of this goodness ( $q$ ) and corresponding  $z$  value obtained from the slope of  $F_2$  as function of  $t$  in log-log scale, besides considering the error bars to determination of slope and the needed error propagation. Here, it is important to mention, that we used the goodness of fit and not the simple and flexible coefficient of determination ( $r$ ) because the error bars were incorporated to obtain quality of the fit, as well as the slope and its uncertainty. In this case,  $z$  is estimated with respective uncertainty as  $\hat{z} \pm \sigma_z = 2/(\widehat{2/z}) \pm 2/(\widehat{2/z})^2 \sigma_{\widehat{2/z}}$ , where  $\widehat{2/z}$  is the slope estimated of  $F_2$  versus  $t$  and  $\sigma_{\widehat{2/z}}$  the error obtained of this fit.

In our algorithm,  $t_{\min}$  varies from 20 up to 300 and  $t_{\max}$  from 80 to 400, with restriction that  $t_{\max} - t_{\min} > 60$  MC steps. As best result, the algorithm supplies  $z = 2.12(2)$ , corresponding to  $q = 0.99999935\dots$  in [300, 360]. We fixed for our analysis 20 points per interval by adjusting the spacing between the points. This value is lower than that

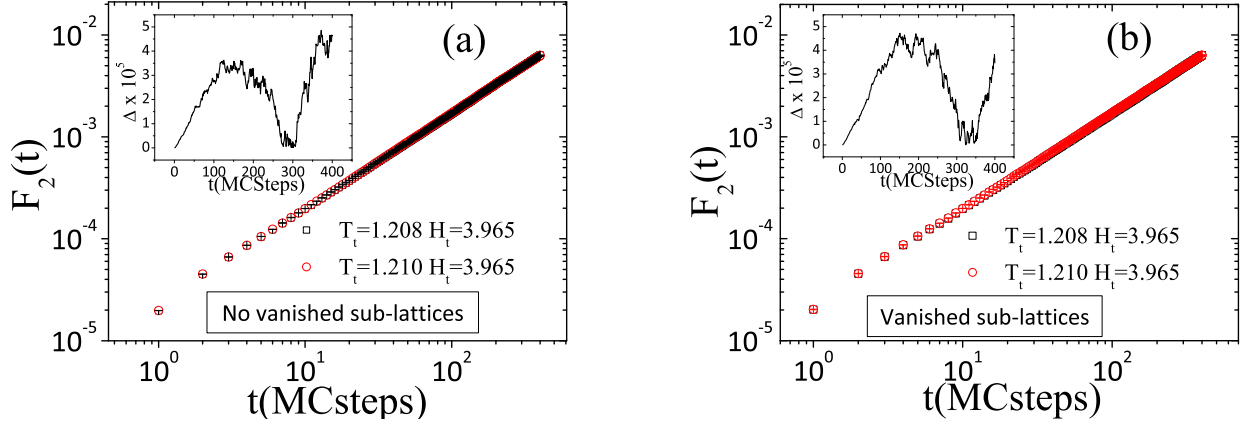


Figure 2: (a): Time evolving in log-log scale of  $F_2(t)$  as function of time  $t$  (corresponding MC step). The square points correspond to simulations that run set on estimate of Landau and Swendsen ( $T_t = 1.208$  and  $H_t = 3.965$ ). The circles corresponds to simulations run set on re-estimated with our refinement ( $T_t = 1.210$  and  $H_t = 3.965$ ). (b): Corresponds to the same plot however the  $m_0 = 0$  is obtained vanishing also the sub-lattices. The same convention is used for both Figure (a) and (b). The inset plots correspond to behavior of difference in absolute value between  $F_2$  obtained with  $T_t = 1.208$  and  $F_2$  obtained with  $T_t = 1.210$ , as function of time. Both cases we used  $H_t = 3.965$ .

estimated for the two-dimensional Blume Capel model at the tricritical point point,  $z = 2.215(2)$  [37].

In order to check if we were exactly on the tricritical point, we decided to reestimate it by using our refinement procedure, as described above. In this case, we fixed  $T_t = 1.210$ , and found  $H_t = 3.965$  (exactly as found in [13]) that is a little bit different of that obtained previously,  $H_t = 3.964$  when  $T_t = 1.208$ .

Then, we performed simulations for  $F_2$  at same conditions but now considering the new set of tricritical parameters,  $T_t = 1.210$  and  $H_t = 3.965$ . The red circles in Figure 2 (a) exhibit such time behavior. We cannot observe a reasonable difference just by looking at the plot. Instead we can take the difference between the two estimates of  $F_2$  (inset Figure 2 (a)) and observe that there is a "microscopic" difference. By performing again our algorithm that finds the best interval of time evolution (corresponding to best goodness of fit) we found  $z = 2.21(2)$ ,  $q = 0.9999938...$  and coincidentally for the same time interval [300, 360]. Such result corroborates our estimate for the Blume-Capel model  $z = 2.215(2)$  [37].

Of course,  $m_0 = 0$  does not imply in sub-lattices with zero magnetization. Although this is an artificial preparation, it is worth to test this situation by observing the time evolution of  $F_2$ , for instance. So, we prepared the initial states with the spin variables at each site chosen at random but with  $M_1(t=0) = M_2(t=0) = m_0 = 0$  in order to study such effects on the exponent  $z$  in comparison with the straight preparation.

With the view to obtain such configurations we performed the following procedure: we randomly selected  $L^2/4$  spins  $\sigma_{i,j}$  with  $i+j$  even, and attributed  $\sigma_{i,j} = 1$ . We also randomly selected  $L^2/4$  spins  $\sigma_{i,j}$  with  $i+j$  odd, and attributed  $\sigma_{i,j} = -1$ . Thereafter, we attribute  $\sigma_{i,j} := -1$  to the remaining spins with  $i+j$  even and  $\sigma_{i,j} := 1$  otherwise. The time evolving under such conditions for the same parameters that was studied without vanishing the sub-lattices can be observed in Fig. 2 (b). Similarly by applying our algorithm to find the best interval, with its corresponding  $z$ , we obtained  $z = 2.13(2)$  in [300, 360] with  $q = 0.9998798...$  for  $T_t = 1.208$  and  $H_t = 3.965$  and  $z = 2.17(3)$  in [320, 380] with  $q = 0.9999955...$ , for  $T_t = 1.210$  and  $H_t = 3.965$ .

We conclude that vanishing the sublattices magnetization seems to be not interesting because the exponent  $z$  presents difference from the natural condition of  $m_0 = 0$ , even though, for  $T_t = 1.210$  and  $H_t = 3.965$ , we find an agreement (according to error bars) of exponents with our best estimate  $z = 2.21(2)$ , obtained without imposing the

Interval	$T_t = 1.208$	$T_t = 1.208$ zero sublattices magnetization	$T_t = 1.210$	$T_t = 1.210$ zero sublattices magnetization
[220,360]	$z = 2.109(8)$ $q = 0.9856$	$z = 2.063(6)$ $q = 0.0885$	$z = 2.177(7)$ $q = 0.9576$	$z = 2.124(7)$ $q = 0.0237$
[240,360]	$z = 2.105(9)$ $q = 0.9993$	$z = 2.086(7)$ $q = 0.6588$	$z = 2.173(9)$ $q = 0.9936$	$z = 2.156(9)$ $q = 0.8355$
[280,360]	$z = 2.12(2)$ $q = 0.9996$	$z = 2.13(2)$ $q = 0.9998$	$z = 2.19(2)$ $q = 0.9773$	$z = 2.19(1)$ $q = 0.9804$
[300,360]	$z = 2.12(3)$ $q = 0.9999$	$z = 2.13(2)$ $q = 0.9999$	$z = 2.21(2)$ $q = 0.9999$	$z = 2.17(3)$ $q = q = 0.9991$

Table 1: Estimates of  $z$  for different intervals and respective goodness of fit found for each time interval analyzed for  $H_t = 3.965$ . Both situations are analyzed with zero sublattices magnetization (no natural choice) and no vanished sub-lattices (natural choice).

vanishing of sub-lattices, whose value is in absolute agreement with estimate to the tricritical point found for the Blume Capel. Table 1 summarizes our main results for  $z$  for the different situations. It is also important to mention that vanishing of the sublattices magnetization, for  $T_t = 1.210$  we can find  $z = 2.21(2)$  for other intervals, for example [280,360] and with goodness  $q = 0.992783\dots$

Since we have determined the tricritical parameters, as well as the tricritical exponent  $z$ , now we can calculate the other tricritical exponents for the metamagnet model, the dynamic critical exponent  $\theta$  and static critical exponents  $\beta$  and  $\nu$ . First, we analyzed the exponent  $\theta$ , that here is calculated by two different methods: i) the straight application of the power law behavior given by Eq. (4) and ii) by means of the time correlation of the order parameter [49].

In the first method, the exponent  $\theta$  is obtained as a function of the initial magnetization  $m_0$ . In this case, it is necessary working with a precise and small value of the initial magnetization in order to obtain  $\theta(m_0)$ . The asymptotic value of  $\theta$  is obtained by extrapolating the estimates of  $\theta$  for various values of  $m_0$  toward the limit  $m_0 \rightarrow 0$ . Our simulations were performed for four different values of  $m_0$ ,  $m_0 = 0.02, 0.04, 0.06$ , and  $0.08$ . Here we used  $L = 160$  and the initial configurations were prepared with fixed  $m_0$  and spins randomly selected following the procedures previously described for  $m_0 = 0$  without vanishing the sublattices. The only difference here is that instead of performing an adjustment to find  $m_0 = 0$  we perform the adjustment to obtain the fixed desired magnetization.

In Figure 3 we showed the behavior of the time evolution of the staggered magnetization for the considered initial magnetizations in double-log scale.

Figure 4 exhibits the behavior of the exponent  $\theta$  for the four initial magnetizations described above, as well as a linear fit that leads to its final value through the numerical extrapolation towards  $m_0 \rightarrow 0$ . The anomalous behavior which prescribes that the magnetization decays as a function of time, instead of an expected increase (as observed in regular critical points) corroborates the numerical observation of the tricritical point of the Blume Capel model [37] as well as the theoretical one [36].

In Table 2 we present the estimates for  $\theta$  as a function of different initial magnetizations  $m_0$ . The value found  $\theta = -0.52(4)$ , from extrapolation to  $m_0 \rightarrow 0$ .

The second method used to estimate  $\theta$  is through the time correlation of the magnetization [49] given by

$$C(t) = \langle M(0)M(t) \rangle \sim t^\theta. \quad (10)$$

When compared to the first technique foreseen by Eq. (4), this method has at least two advantages. It does not demand a careful preparation of the initial configurations the limiting procedure, the only requirement being that  $\langle m_0 \rangle = 0$ . From a computational point of view it is very useful, because the spins are placed randomly on the lattice sites and the



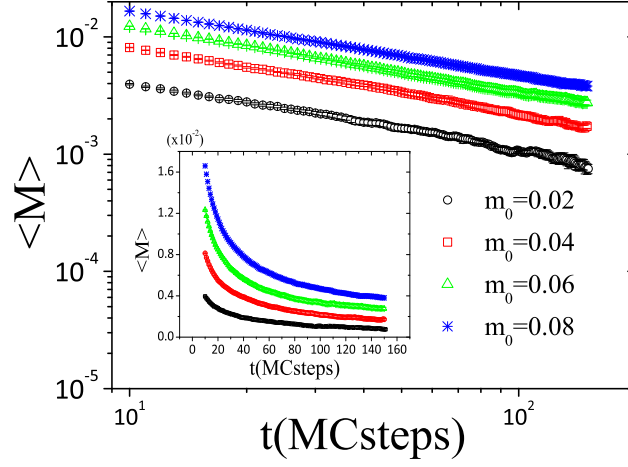


Figure 3: Time evolution of the magnetization for  $m_0 = 0.02, 0.04, 0.06$  and  $0.08$ . The error bars were calculated over 5 sets of 20000 runs each one. The inset displays the time evolution in a linear scale.

$m_0$	$\theta$
0.08	$-0.585(2)$
0.06	$-0.576(5)$
0.04	$-0.566(1)$
0.02	$-0.55(1)$
Extrapolated value	$-0.52(4)$

Table 2: The dynamical exponent  $\theta$  from the time evolution of the magnetization for different initial configurations.

evolution starts without questions about the value of  $m_0$  since that procedure ensures magnetizations around  $m_0 = 0$ .

Figure 5 displays the time dependence of the time correlation  $C(t)$  in double-log scale. The linear fit of this curve leads to the value

$$\theta = -0.56(2). \quad (11)$$

These results [see Table 2 and Eq. (11)] are in agreement with the value obtained for the Blume-Capel model [37] at the tricritical point,  $\theta = -0.53(2)$ , corroborating the dynamical universality for the tricritical points, as well as confirming the conjecture by Janssen and Oerding [36] whereas the value of  $\theta$  for the metamagnetic model is also negative.

Let us consider now the static critical exponents  $\nu$  and  $\beta$  of the metamagnetic model, both obtained through the scaling behavior of the staggered magnetization and taking into account runs with ordered initial configurations ( $m_0 = 1$ ). The static exponent  $\nu$  can be obtained by fixing  $b^{-z}t = 1$  in Eq. (3) and differentiating  $\ln M(t, \tau)$  with respect to  $\tau$  at the tricritical point. The power law obtained is

$$D(t) = \frac{\partial}{\partial \tau} \ln \langle M \rangle_{m_0=1}(t, \tau)|_{\tau=0} \sim t^{1/\nu z}. \quad (12)$$

Numerically, the quantity  $D(t)$  is computed simulating the relaxation of the system initially ordered in two different

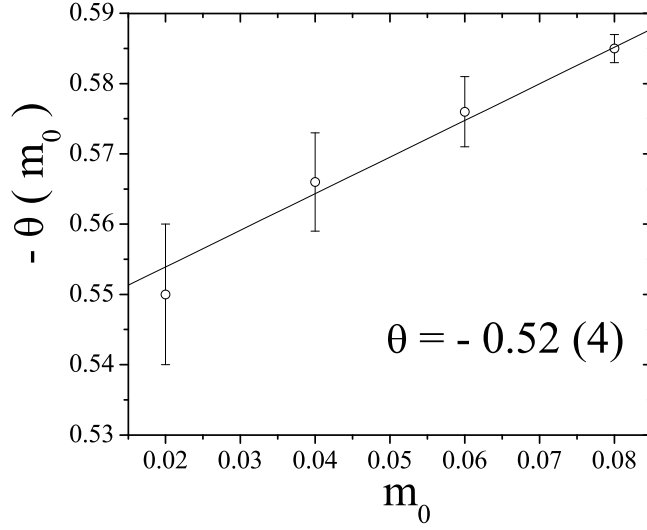


Figure 4: Dynamic exponent  $\theta$  as a function of the initial magnetization  $m_0$ . Each point represents an average over 5 sets of 20000 runs each one.

points, the first one slightly above the tricritical temperature ( $T_t + \varepsilon$ ) and the other one slightly below the tricritical temperature ( $T_t - \varepsilon$ ), keeping the magnetic field  $H_t$  fixed. So, we numerically expected that

$$D(t) = \frac{1}{2\varepsilon} \ln \frac{\langle M \rangle_{m_0=1}(t, \tau + \varepsilon)}{\langle M \rangle_{m_0=1}(t, \tau - \varepsilon)} \sim t^{1/\nu z}$$

for small value of  $\varepsilon$ . In previous works we typically used  $\varepsilon = O(10^{-3})$  and in this work we considered  $\varepsilon = 1 \cdot 10^{-3}$ . In Figure 6, the power law increase of the above equation is plotted in double-log scale.

From the slope of the curve one can estimate the tricritical exponent  $1/\nu z$  and by using the exponent  $z$  obtained from the scaling relation  $F_2(t)$ , the exponent  $\nu$  is then estimated with its respective uncertainty (error propagated):

$$\hat{\nu} \pm \sigma_\nu = \frac{\left( \widehat{1/\nu z} \cdot \hat{z} \right)^{-1} \pm \sqrt{\left( \widehat{1/\nu z}^2 \cdot \hat{z} \right)^{-2} \sigma_{\widehat{1/\nu z}}^2 + \left( \widehat{1/\nu z} \cdot \hat{z}^2 \right)^{-2} \sigma_{\hat{z}}^2}}{\quad}$$

Here  $\widehat{1/\nu z}$  and  $\hat{z}$  are the estimates and  $\sigma_{\widehat{1/\nu z}}$  and  $\sigma_{\hat{z}}$  are their respective uncertainties. Our estimate for  $\nu$  at the tricritical point is  $\nu_t = 0.57(3)$  at interval  $[320, 380]$  with goodness-of-fit  $q = 1$  which corroborates the theoretical prediction  $\nu_t = 5/9 = 0.55\bar{5}$ . Here we kept 20 points per interval and used the same processing to find the best goodness. Here it is also important to mention that once we have simulated 5 different bins for  $T_t - \varepsilon$  ( $T = 1.209$ ) and 5 bins for  $T_t + \varepsilon$  ( $T = 1.211$ ) and by crossing all seeds, we obtained a sample with 25 different measures as well as the procedure used for  $F_2$ , i.e., in the first case we have crossed seeds of different temperatures and in the second case due to the different initial conditions to compose  $F_2$ .

Finally, we evaluated the statical exponent  $\beta$  by the dynamic scaling law for the magnetization  $\langle M \rangle_{m_0=1}(t) \sim t^{-\beta/\nu z}$ . By estimating  $\widehat{\beta/\nu z}$  from the log-log plot of  $\langle M \rangle$  versus  $t$ ,  $\beta$  is determined with its uncertainty as

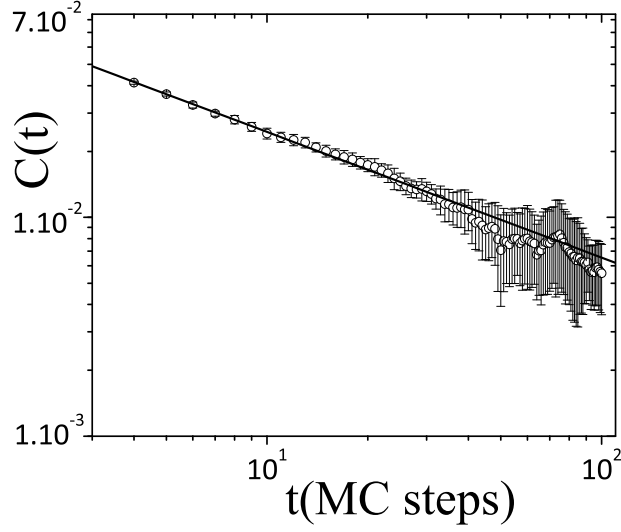


Figure 5: Time correlation of the magnetization for samples with  $\langle M(t=0) \rangle \approx 0$ . The error bars were calculated over 5 sets of 20000 runs each one.

Exponent	ref. [37] 2D Blume Capel (short time dynamics)	ref. [14] Quantum 1d BEG Model (FSS)	ref. [20] Blume Capel model (MCRG)	ref. [12] (FSS)	ref. [13] (FSS)	ref. [50, 51] (CI)	This work
$\beta$	0.0453(2)	—	0.039	0.0411(7)	—	1/24	0.049(4)
$\nu$	0.537(6)	1/1.80	0.56	0.552(6)	0.5562(12)*	5/9	0.57(3)

Table 3: Static tricritical exponents. We present some results found in the literature and our predictions via time-dependent MC simulations. The acronyms (BEG), (MCRG), (FSS) and (CI) mean "Blume-Emery-Griffiths", "Monte Carlo "Renormalization Group", Finite Size Scalling, and Conformal Invariance respectively.

$$\hat{\beta} \pm \sigma_{\beta} = \left( \widehat{\beta/\nu z} \right) \cdot \left( \widehat{1/\nu z} \right)^{-1} \pm \sqrt{\left( \widehat{1/\nu z} \right)^{-2} \sigma_{\widehat{\beta/\nu z}}^2 + \left( \widehat{\beta/\nu z} \right) \cdot \left( \widehat{1/\nu z} \right)^{-2} \sigma_{\widehat{1/\nu z}}^2}$$

In Figure 7 we show the time evolution of the magnetization in double-log scale.

The exponent obtained from the slope of this curve is  $\beta/\nu z = 0.0390(2)$  in the time interval  $[320, 380]$  with  $q = 0.9999\dots$ . With this exponent in hand and taking into account the previous result for  $1/\nu z$  ( $0.79(7)$ ), we can estimate the exponent  $\beta$  through the equation above. The result,  $\beta = 0.049(4)$ , is close to the theoretical prediction  $\beta = 1/24$ . In Table 3 we show our estimates and a comparison with important results from literature of statical exponents ( $\beta$  and  $\nu$ ) at the tricritical points.

In summary, we have performed short-time Monte Carlo simulations to investigate the scaling behavior at the tricritical point of a two-dimensional metamagnetic model. The dynamic critical exponent  $\theta$  was estimated using two different approaches: by following the time evolution of the staggered magnetization and measuring the evolution of

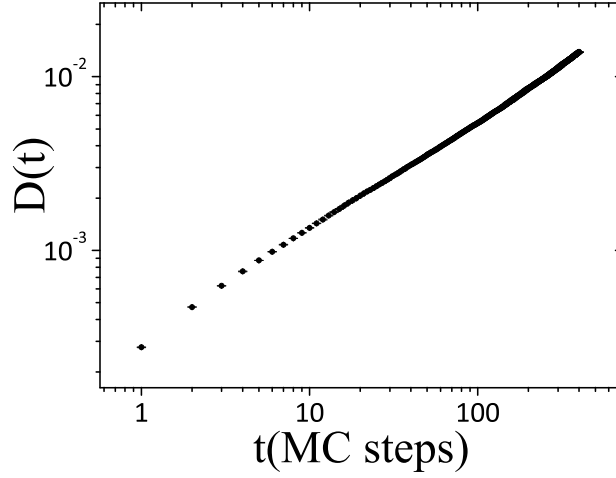


Figure 6: The time evolution of the derivative  $\partial M(t, \tau)/\partial \tau|_{\tau=0}$  in log-log scale in a dynamic process starting from an ordered state ( $m_0 = 1$ ). Error bars are smaller than the symbols. Each point represents an average over 25 sets (5 at  $T_c - \varepsilon$  crossed with 5 at  $T_c + \varepsilon$ ) of 10000 runs each one.

the time correlation function  $C(t)$  of the staggered magnetization. On the other hand, the dynamic critical exponent  $z$  was found through the function  $F_2(t)$  which combines simulations performed with different initial conditions. The static critical exponents  $\beta$  and  $\nu$  were obtained through the scaling relations for the staggered magnetization and its derivative with respect to the temperature at  $T_c$ . Our results are in good agreement with the exponents previously determined for the tricritical point of the two-dimensional Blume-Capel model.

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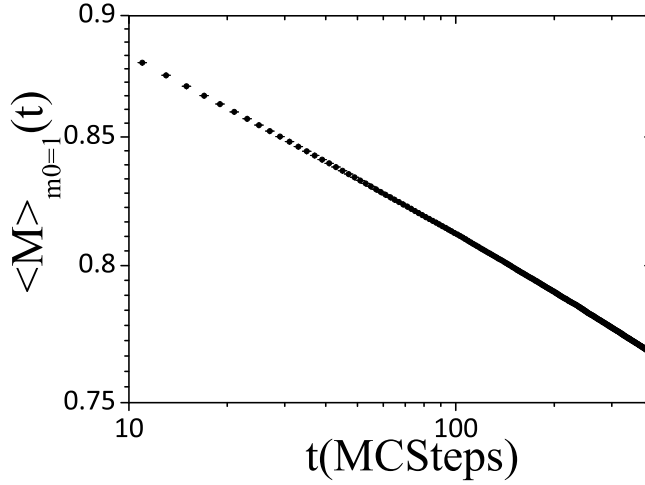


Figure 7: The time evolution of the magnetization for initially ordered samples ( $m_0 = 1$ ). The error bars calculated over 5 sets of 10000 runs each one, are smaller than the symbols.

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